

**PT-/NON-PT-SYMMETRIC AND NON-HERMITIAN HELLMANN POTENTIAL:****APPROXIMATE BOUND AND SCATTERING STATES WITH ANY  $\ell$ -VALUES**

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*We investigate the approximate bound state solutions of the Schrödinger equation for the PT-/non-PT-symmetric and non Hermitian Hellmann potential. Exact energy eigenvalues and corresponding normalized wave functions are obtained. Numerical values of energy eigenvalues for the bound states are compared with the ones obtained before. Scattering state solutions are also studied. Phase shifts of the potential are written in terms of the angular momentum quantum number  $\ell$ .*

**1 INTRODUCTION**

PT-symmetric quantum mechanics has been widely studied in recent years. The usual form of quantum mechanics has the Hamiltonian defining symmetries of the system is Hermitian. However for PT-symmetric case, it has real spectra although it is not Hermitian. Bender and Boettcher [1] studied this case. Later many authors studied PT-symmetric and non-Hermitian cases having real and/or complex eigenvalues [2–10].

The Hellmann potential

$$V(r) = \frac{1}{r} \left( -a + be^{-\lambda r} \right), \quad (1)$$

with  $b > 0$  was first proposed by Hellmann [11, 12] (then called as the 'Hellmann potential' independently of the sign of  $b$ ) which has many applications in atomic physics and condensed physics [13]. The Hellmann potential has been used as a potential model to calculate the electronic wave functions of metals and semiconductors [14]. Many authors have studied the electron-core [15–17] and electron-ion [8] interactions by using this potential. In Ref. [19], it has been proposed that the Hellmann potential is a suitable ground for study of inner-shell ionization problems. The present potential could be used as a potential model for the alkali hydride molecules [20].

Energy eigenvalues of the Hellmann potential have recently been studied by various authors with the help of different methods such as  $1/N$  expansion method [21], shifted large- $N$  expansion method [22], the method of potential envelopes [23], the  $J$ -matrix approach [24] and the generalized Nikiforov-Uvarov method [25, 26]. In the present work, we solve the Schrödinger-Hellmann problem in terms of the hypergeometric functions by using an approximation

instead of the centrifugal term. We extend also the computation including the solutions of the Hellmann-like potential having the form

$$V(x) = -\frac{a}{x} + \frac{b}{x} e^{-\lambda x}, \quad (2)$$

which can be written in a PT-symmetric form and the energy spectra and eigenfunctions of PT-/non-PT- and non-Hermitian Hellmann potential are obtained with any angular momentum. Scattering state solutions are also studied. Phase shifts of the potential are written in terms of the angular momentum quantum number  $\ell$ .

The organization of this work is as follows. In Section 2, we find the approximate energy eigenvalues and the corresponding normalized wave functions of the Hellmann potential. In Section 3, we obtain the phase shifts of the potential under consideration in terms of the quantum number  $\ell$ . We give our conclusions in last section.

## 2 BOUND STATES

### i. Radial Solutions

The radial part of the Schrödinger equation (SE) [27]

$$\left\{ \frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + \frac{2m}{\hbar^2} [E - V(r)] \right\} \mathcal{R}(r) = 0. \quad (3)$$

where  $\mathcal{R}(r) = R(r)/r$ ,  $\ell$  is the angular momentum quantum number,  $m$  is the particle mass moving in the potential field  $V(r)$  and  $E$  is the nonrelativistic energy of particle.

Using the following approximation instead of the centrifugal term [28]

$$\frac{1}{r^2} \sim \frac{\lambda^2}{(1 - e^{-\lambda r})^2}, \quad (4)$$

inserting Eq. (1) into Eq. (3) and defining a new variable  $u = e^{-\lambda r}$  ( $0 \leq u \leq 1$ ), Eq. (3) becomes

$$\begin{aligned} & u(1-u) \frac{d^2 R(u)}{du^2} + (1-u) \frac{dR(u)}{du} \\ & \times \left[ -\frac{\ell(\ell+1)}{1-u} + \left( \frac{2mE}{\lambda^2 \hbar^2} + \frac{2ma\lambda}{\lambda^2 \hbar^2} - \ell(\ell+1) \right) \frac{1}{u} - \frac{2mb\lambda}{\lambda^2 \hbar^2} - \frac{2mE}{\lambda^2 \hbar^2} \right] R(u) = 0. \end{aligned} \quad (5)$$

Taking a trial wave function as

$$R(u) = u^{\lambda_1} (1-u)^{\lambda_2} \psi(u), \quad (6)$$

and inserting into Eq. (5), we obtain

$$u(1-u)\frac{d^2\psi(u)}{du^2} + [1 + 2\lambda_1 - (2\lambda_1 + 2\lambda_2 + 1)u] \frac{d\psi(u)}{du} \\ \times \left[ -\lambda_1^2 - \lambda_2^2 - 2\lambda_1\lambda_2 - \frac{2m}{\lambda^2\hbar^2}(E + b\lambda) \right] \psi(u) = 0, \quad (7)$$

where

$$\lambda_1^2 = -\frac{2m}{\lambda^2\hbar^2}(E + a\lambda) + \ell(\ell + 1), \quad (8)$$

$$\lambda_2 = \frac{1}{2} \left[ 1 \pm \sqrt{1 + 4\ell(\ell + 1)} \right]. \quad (9)$$

Comparing Eq. (7) with the hypergeometric equation of the following form [29]

$$u(1-u)\frac{d^2y(u)}{du^2} + [c' - (a' + b' + 1)u] \frac{dy(u)}{du} - a'b'y(u) = 0, \quad (10)$$

we find the solution of Eq. (7) as the hypergeometric function

$$\psi(u) = {}_2F_1(a', b'; c'; u), \quad (11)$$

with

$$a' = \lambda_1 + \lambda_2 + \Lambda_1, \quad (12)$$

$$b' = \lambda_1 + \lambda_2 - \Lambda_1, \quad (13)$$

$$c' = 1 + 2\lambda_1. \quad (14)$$

where  $\Lambda_1 = \frac{1}{2}\sqrt{-\frac{8m}{\lambda^2\hbar^2}(E + b\lambda)}$ .

The total wave functions in Eq. (6) are given as

$$R(u) = \mathcal{N}u^{\lambda_1}(1-u)^{\lambda_2} {}_2F_1(a', b'; c'; u). \quad (15)$$

where  $\mathcal{N}$  is normalization constant and will be calculated below. When either  $a'$  or  $b'$  equals to a negative integer  $-n$ , the hypergeometric function  $\psi(u)$  can give a finite solution form. This gives us a polynomial of degree  $n$  in Eq. (11) and from the following quantum condition

$$-n = \lambda_1 + \lambda_2 + \frac{1}{2}\sqrt{-\frac{8m}{\lambda^2\hbar^2}(E + b\lambda)}, \quad (n = 0, 1, 2, \dots) \quad (16)$$

the energy eigenvalue becomes

$$E = -\frac{1}{8m\hbar^2(n + \ell + 1)^2} \left\{ 4m^2(a^2 + b^2) + 4m\hbar^2\lambda b [2\ell^2 + (n + \ell)^2 + \ell(3 + 2n)] \right. \\ \left. + \lambda^2\hbar^4 [\ell(1 + 2n) + (n + \ell)^2]^2 + 4am [-2bm + \lambda\hbar^2 [\ell(1 + 2n) + (n + \ell)^2]] \right\}. \quad (17)$$

The numerical results obtained from last equation are listed in Table I. We also compare them with the ones given in two different papers. They are in agreement with those of the previous results where we should also stress that our results are consistent with the ones given in Ref. [25]. Eq. (16) gives the wave functions as

$$R(u) = \mathcal{N} u^{\lambda_1} (1-u)^{\lambda_2} {}_2F_1(-n, n+2\lambda_1+2\lambda_2; 1+2\lambda_1; u), \quad (18)$$

where the normalization constant is obtained from  $\int_0^1 |R(u)|^2 du = 1$  which can be written as

$$|\mathcal{N}|^2 \frac{\Gamma(1+2\lambda_1)}{m! \Gamma(-n) \Gamma(n+2\lambda_1+2\lambda_2)} \sum_{m=0}^{\infty} \frac{\Gamma(-n+m) \Gamma(n+2\lambda_1+2\lambda_2+m)}{\Gamma(1+2\lambda_1+m)} \\ \times \int_0^1 u^{m+2\lambda_1} (1-u)^{2\lambda_2} {}_2F_1(-n, n+2\lambda_1+2\lambda_2; 1+2\lambda_1; u) du = 1. \quad (19)$$

We use the following representation of the hypergeometric functions [29]

$${}_2F_1(p, q; r; z) = \frac{\Gamma(r)}{\Gamma(p) \Gamma(q)} \sum_{m=0}^{\infty} \frac{\Gamma(p+m) \Gamma(q+m)}{\Gamma(r+m)} \frac{z^m}{m!}, \quad (20)$$

By using the following identity [30]

$$\int_0^1 s^{\nu-1} (1-s)^{\mu-1} {}_2F_1(\alpha, \beta; \gamma; as) ds = \frac{\Gamma(\mu) \Gamma(\nu)}{\Gamma(\mu+\nu)} {}_3F_2(\nu, \alpha, \beta; \mu+\nu; \gamma; a), \quad (21)$$

we obtain the normalization constant as

$$|\mathcal{N}|^2 = \frac{\Gamma(1+2\lambda_1) \Gamma(1+2\lambda_2)}{m! \Gamma(-n) \Gamma(n+2\lambda_1+2\lambda_2)} \sum_{m=0}^{\infty} \frac{\Gamma(-n+m) \Gamma(n+2\lambda_1+2\lambda_2+m)}{\Gamma(1+2\lambda_1+m)} \\ \times {}_3F_2(1+2\lambda_1+m, -n, n+2\lambda_1+2\lambda_2; 2+2\lambda_1+2\lambda_2+m; 1+2\lambda_1; 1). \quad (22)$$

For the completeness, we give the energy eigenvalues of the Coulomb potential which corresponds to the case where  $b = 0$ . We write the energy levels of this potential from Eq. (17) as ( $\lambda = 0, \hbar = 1$ )

$$E = -\frac{ma^2}{2(n+\ell+1)^2}. \quad (23)$$

## ii. PT-Symmetric Solutions

Inserting Eq. (2) into the following one-dimensional SE [27]

$$-\frac{\hbar^2}{2m} \frac{d^2 \phi(x)}{dx^2} + [V(x) - E] \phi(x) = 0. \quad (24)$$

using the following approximation instead of  $1/x$  in the potential (see, Fig. 1)

$$\frac{1}{x} \sim \frac{\lambda}{1 - e^{-\lambda x}}, \quad (25)$$

and taking a new variable as  $u = 1/(1 - e^{-\lambda x})$  we obtain

$$\begin{aligned} & u(1-u) \frac{d^2 \phi(u)}{du^2} + (1-2u) \frac{d\phi(u)}{du} \\ & \times \left[ \left( \frac{2ma}{\lambda \hbar^2} + \frac{2mE}{\lambda^2 \hbar^2} \right) \frac{1}{1-u} + \left( \frac{2mb}{\lambda \hbar^2} + \frac{2mE}{\lambda^2 \hbar^2} \right) \frac{1}{u} \right] \phi(u) = 0. \end{aligned} \quad (26)$$

Defining the wave function as

$$\phi(u) = u^{\lambda_1} (1-u)^{\lambda_2} \psi(u), \quad (27)$$

and following the same procedure in the above we get

$$\begin{aligned} & u(1-u) \frac{d^2 \psi(u)}{du^2} + [1 + 2\lambda_1 - (2\lambda_1 + 2\lambda_2 + 1)u] \frac{d\psi(u)}{du} \\ & - [\lambda_1(\lambda_1 + 1) + \lambda_2(\lambda_2 + 1) + 2\lambda_1\lambda_2] \psi(u) = 0, \end{aligned} \quad (28)$$

where

$$\lambda_1^2 = -\frac{2m}{\lambda \hbar^2} \left( b + \frac{E}{\lambda} \right), \quad (29)$$

$$\lambda_2^2 = -\frac{2m}{\lambda \hbar^2} \left( a + \frac{E}{\lambda} \right). \quad (30)$$

The solution of Eq. (28) and the total function is given as, respectively,

$$\psi(u) \sim {}_2F_1(a', b'; c'; u), \quad (31)$$

$$\phi(u) \sim u^{\lambda_1} (1-u)^{\lambda_2} {}_2F_1(a', b'; c'; u), \quad (32)$$

where

$$a' = 1 + \lambda_1 + \lambda_2, \quad (33)$$

$$b' = \lambda_1 + \lambda_2, \quad (34)$$

$$c' = 1 + 2\lambda_1. \quad (35)$$

The energy eigenvalues are written as

$$E = -\frac{\lambda}{4A(1+n)^2} \left\{ a^2 A^2 + 2aA[-Ab + (1+n)^2] + [Ab + (1+n)^2]^2 \right\}. \quad (36)$$

where  $A = 2m/\lambda \hbar^2$ .

iii. *Non-Hermitian PT-Symmetric Form*

Changing the potential parameters as  $a \rightarrow ia, b \rightarrow ib, \beta \rightarrow i\beta$  in Eq. (2), the potential satisfies

$$V^*(-x) = \left( \frac{ia}{x} - \frac{ib}{x} e^{i\beta x} \right)^* = V(x), \quad (37)$$

which shows that we obtain non-Hermitian PT-symmetric form of the Hellmann-like potential. Inserting Eq. (37) into Eq. (24) and using the variable  $u = [1 - e^{-i\lambda x}]^{-1}$ , we obtain

$$u(1-u) \frac{d^2\phi(u)}{du^2} + (1-2u) \frac{d\phi(u)}{du} \times \left[ \left( \frac{2ma}{\lambda\hbar^2} - \frac{2mE}{\lambda^2\hbar^2} \right) \frac{1}{1-u} + \left( \frac{2mb}{\lambda\hbar^2} - \frac{2mE}{\lambda^2\hbar^2} \right) \frac{1}{u} \right] \phi(u) = 0. \quad (38)$$

where

$$\lambda_1^2 = -\frac{2m}{\lambda\hbar^2} \left( b - \frac{E}{\lambda} \right), \quad (39)$$

$$\lambda_2^2 = -\frac{2m}{\lambda\hbar^2} \left( a - \frac{E}{\lambda} \right). \quad (40)$$

Following the same steps, we obtain the wave function for the non-Hermitian PT-symmetric Hellmann-like potential

$$\phi(u) \sim u^{\lambda_1}(1-u)^{\lambda_2} {}_2F_1(a', b'; c'; u), \quad (41)$$

and the energy spectrum as

$$E = -\frac{1}{8m\hbar^2(1+n)^2} \left\{ 4m^2a^2 + 4ma[-2mb + \lambda\hbar^2(1+n)^2] + [2mb + \lambda\hbar^2(1+n)^2]^2 \right\}. \quad (42)$$

#### iv. Non-Hermitian Non-PT-Symmetric Form

Case 1:  $a$  and  $b$  real,  $\lambda \rightarrow i\lambda$

In this case the potential satisfies  $[V(x)]^* \neq V(x)$  so it has a non-Hermitian non-PT-symmetric form given as

$$V(x) = -ia\lambda \frac{1}{1 - e^{-i\lambda x}} + ib\lambda \frac{e^{-i\lambda x}}{1 - e^{-i\lambda x}}, \quad (43)$$

Using the variable  $u = [1 - e^{-i\lambda x}]^{-1}$ , we obtain

$$u(1-u) \frac{d^2\phi(u)}{du^2} + (1-2u) \frac{d\phi(u)}{du} \times \left[ -\left( \frac{2mia}{\lambda\hbar^2} + \frac{2mE}{\lambda^2\hbar^2} \right) \frac{1}{1-u} - \left( \frac{2mb}{\lambda\hbar^2} + \frac{2mE}{\lambda^2\hbar^2} \right) \frac{1}{u} \right] \phi(u) = 0. \quad (44)$$

with

$$\lambda_1^2 = \frac{2m}{\lambda\hbar^2} \left( ib + \frac{E}{\lambda} \right), \quad (45)$$

$$\lambda_2^2 = \frac{2m}{\lambda\hbar^2} \left( ia + \frac{E}{\lambda} \right). \quad (46)$$

Following the same steps, we obtain the wave function for the non-Hermitian non-PT-symmetric Hellmann-like potential

$$\phi(u) \sim u^{\lambda_1} (1-u)^{\lambda_2} {}_2F_1(a', b'; c'; u), \quad (47)$$

and the energy spectrum as

$$E = -\frac{1}{8m\hbar^2(1+n)^2} \left\{ 4m^2 a^2 - 4ma[2mb - i\lambda\hbar^2(1+n)^2] + [2mb + i\lambda\hbar^2(1+n)^2]^2 \right\}. \quad (48)$$

Case 2:  $\beta$  real,  $a \rightarrow ia$  and  $b \rightarrow ib$

By using the variable  $t = 1/(1-e^{-\beta x})$  we obtain the following in the present case

$$V(x) = -ia\lambda \frac{1}{1-e^{-\lambda x}} + ib\lambda \frac{e^{-\lambda x}}{1-e^{-\lambda x}}, \quad (49)$$

Using the variable  $u = [1 - e^{-\lambda x}]^{-1}$ , we obtain

$$u(1-u) \frac{d^2 \phi(u)}{du^2} + (1-2u) \frac{d\phi(u)}{du} \times \left[ \left( \frac{2mia}{\lambda\hbar^2} + \frac{2mE}{\lambda^2\hbar^2} \right) \frac{1}{1-u} + \left( \frac{2mib}{\lambda\hbar^2} + \frac{2mE}{\lambda^2\hbar^2} \right) \frac{1}{u} \right] \phi(u) = 0. \quad (50)$$

where

$$\lambda_1^2 = -\frac{2m}{\lambda\hbar^2} \left( ib + \frac{E}{\lambda} \right), \quad (51)$$

$$\lambda_2^2 = -\frac{2m}{\lambda\hbar^2} \left( ia + \frac{E}{\lambda} \right). \quad (52)$$

Following the same steps as in the above section we obtain the wave function as

$$\phi(u) \sim u^{\lambda_1} (1-u)^{\lambda_2} {}_2F_1(a', b'; c'; u), \quad (53)$$

and the energy spectrum as

$$E = \frac{1}{8m\hbar^2(1+n)^2} \left\{ 4m^2a^2 - 4ma[2mb + i\lambda\hbar^2(1+n)^2] + [2mb - i\lambda\hbar^2(1+n)^2]^2 \right\}. \quad (54)$$

We now study the scattering state solutions of the Hellmann potential in the next section.

### 3 SCATTERING STATES AND PHASE SHIFTS

In order to obtain the scattering state solutions we choose the variable as  $t = 1 - e^{-\lambda r}$  in Eq. (3) and we get the following equation in terms of  $t$

$$t(1-t)\frac{d^2R(t)}{dt^2} - t\frac{dR(t)}{dt} + \left[ \left[ \frac{2m}{\lambda\hbar^2}(a - \epsilon) - \ell(\ell+1) \right] \frac{1}{1-t} - \ell(\ell+1)\frac{1}{t} - \frac{2m}{\lambda\hbar^2}(b - \epsilon) \right] R(t) = 0, \quad (55)$$

where  $-\epsilon = E/\lambda$ . By using a trial wave function as  $R(t) = t^\mu(1-t)^\nu\psi(t)$ , we get

$$t(1-t)\frac{d^2\psi(t)}{dt^2} + [2\mu - (2\mu + 2\nu + 1)t]\frac{d\psi(t)}{dt} + \left[ \mu^2 + \nu^2 + 2\mu\nu + \frac{2m}{\lambda\hbar^2}(b - \epsilon) \right] \psi(t) = 0, \quad (56)$$

with

$$\mu = \begin{cases} -\ell \\ 1 + \ell \end{cases} ; \nu = -i\kappa ; \kappa = \sqrt{\frac{2m}{\lambda\hbar^2}(a - \epsilon) - \ell(\ell+1)}. \quad (57)$$

By using the following abbreviations

$$\xi_1 = \mu - i\kappa + \Lambda_2, \quad (58)$$

$$\xi_2 = \mu - i\kappa - \Lambda_2, \quad (59)$$

$$\xi_3 = 2\mu, \quad (60)$$

where  $\Lambda_2 = \sqrt{\frac{2m}{\lambda\hbar^2}(\epsilon - b)}$ , Eq. (56) can be written as a hypergeometric-type equation [30]

$$t(1-t)\frac{d^2\psi(t)}{dt^2} + [\xi_3 - (\xi_1 + \xi_2 + 1)t]\frac{d\psi(t)}{dt} - \xi_1\xi_2\psi(t) = 0. \quad (61)$$



Its solution is given by

$$\psi(t) = {}_2F_1(\xi_1, \xi_2; \xi_3; t), \quad (62)$$

where the parameters satisfy the followings

$$\xi_3 - \xi_1 - \xi_2 = (\xi_1 + \xi_2 - \xi_3)^*; \quad \xi_3 - \xi_1 = \xi_2^*; \quad \xi_3 - \xi_2 = \xi_1^*. \quad (63)$$

which are used in determination of the phase shifts. By using the following equality of the hypergeometric functions [30]

$$\begin{aligned} {}_2F_1(a'', b''; c''; z) &= \frac{\Gamma(c'')\Gamma(c'' - a'' - b'')}{\Gamma(c'' - a'')\Gamma(c'' - b'')} {}_2F_1(a'', b''; a'' + b'' - c'' + 1; 1 - z) \\ &+ (1 - z)^{c'' - a'' - b''} \times \\ &\frac{\Gamma(c'')\Gamma(a'' + b'' - c'')}{\Gamma(a'')\Gamma(b'')} {}_2F_1(c'' - a'', c'' - b''; c'' - a'' - b'' + 1; 1 - z), \end{aligned} \quad (64)$$

and  ${}_2F_1(a'', b''; c''; 0) = 1$ , we write the solution of Eq. (61) in the limit of  $r \rightarrow \infty$  as

$$\begin{aligned} &{}_2F_1(\xi_1, \xi_2; \xi_3; 1 - e^{-\lambda r}) \xrightarrow{r \rightarrow \infty} \\ &\frac{\Gamma(2\mu)\Gamma(2i\kappa)}{\Gamma\left(\mu + i\kappa - \sqrt{\frac{2m}{\lambda\hbar^2}}(\epsilon - b)\right)\Gamma\left(\mu + i\kappa + \sqrt{\frac{2m}{\lambda\hbar^2}}(\epsilon - b)\right)} \\ &+ e^{-2i\kappa\lambda r} \frac{\Gamma(2\mu)\Gamma(-2i\kappa)}{\Gamma\left(\mu - i\kappa + \sqrt{\frac{2m}{\lambda\hbar^2}}(\epsilon - b)\right)\Gamma\left(\mu - i\kappa - \sqrt{\frac{2m}{\lambda\hbar^2}}(\epsilon - b)\right)}. \end{aligned} \quad (65)$$

Defining the following

$$\frac{\Gamma(c'' - a'' - b'')}{\Gamma(c'' - a'')\Gamma(c'' - b'')} = \left| \frac{\Gamma(c'' - a'' - b'')}{\Gamma(c'' - a'')\Gamma(c'' - b'')} \right| e^{i\delta}, \quad (66)$$

and also with the help of Eq. (63)

$$\left( \frac{\Gamma(c'' - a'' - b'')}{\Gamma(c'' - a'')\Gamma(c'' - b'')} \right)^* = \left| \frac{\Gamma(c'' - a'' - b'')}{\Gamma(c'' - a'')\Gamma(c'' - b'')} \right| e^{-i\delta}, \quad (67)$$

Eq. (65) becomes

$$\begin{aligned} &{}_2F_1(\xi_1, \xi_2; \xi_3; 1 - e^{-\lambda r}) \xrightarrow{r \rightarrow \infty} \\ &\Gamma(2\mu) \left| \frac{\Gamma(2i\kappa)}{\Gamma\left(\mu + i\kappa - \sqrt{\frac{2m}{\lambda\hbar^2}}(\epsilon - b)\right)\Gamma\left(\mu + i\kappa + \sqrt{\frac{2m}{\lambda\hbar^2}}(\epsilon - b)\right)} \right| \\ &\times e^{-i\kappa\lambda r} \left[ e^{i(\delta + \kappa\lambda r)} + e^{-i(\delta + \kappa\lambda r)} \right]. \end{aligned} \quad (68)$$

We write the total wave function with the help of this result as

$$R(r \rightarrow \infty) = 2\Gamma(2\mu) \left| \frac{\Gamma(2i\kappa)}{\Gamma\left(\mu + i\kappa - \sqrt{\frac{2m}{\lambda\hbar^2}}(\epsilon - b)\right) \Gamma\left(\mu + i\kappa + \sqrt{\frac{2m}{\lambda\hbar^2}}(\epsilon - b)\right)} \right| \times \sin\left(\delta + \lambda\kappa r + \frac{\pi}{2}\right), \quad (69)$$

Comparing this result with the boundary condition of the scattering state wave function as  $u(r \rightarrow \infty) \rightarrow 2\sin\left(kr - \frac{\pi}{2}\ell + \delta_\ell\right)$ , we obtain the phase shifts as

$$\begin{aligned} \delta_\ell &= \frac{\pi}{2}(1 + \ell) + \arg\Gamma(2i\kappa) - \arg\Gamma\left(\mu + i\kappa - \sqrt{\frac{2m}{\lambda\hbar^2}}(\epsilon - b)\right) \\ &\quad - \arg\Gamma\left(\mu + i\kappa + \sqrt{\frac{2m}{\lambda\hbar^2}}(\epsilon - b)\right). \end{aligned} \quad (70)$$

It is seen that the phase shifts of the Hellmann potential can be produced by using the behavior of the hypergeometric functions at infinity and they are dependent on the energy of the particle.

## 4 CONCLUSION

We have solved the Schrödinger equation for PT-/non-PT-symmetric and non-Hermitian Hellmann potential for any angular momentum. The normalized wave functions are obtained in terms of the hypergeometric functions by using an approximation instead of the centrifugal term. We have calculated energy eigenvalue. Its numerical values for the bound states are listed in Table I. They are compared with those of the previous results. We have seen that our results are in good agreement especially for smaller parameter values. The energy eigenvalue for the Coulomb potential is obtained by setting potential parameters. Finally, we have studied the scattering state solutions of the Hellmann potential and obtained the phase shifts in terms of the angular momentum quantum number  $\ell$ .

## 5 ACKNOWLEDGMENTS

This research was partially supported by the Scientific and Technical Research Council of Turkey.

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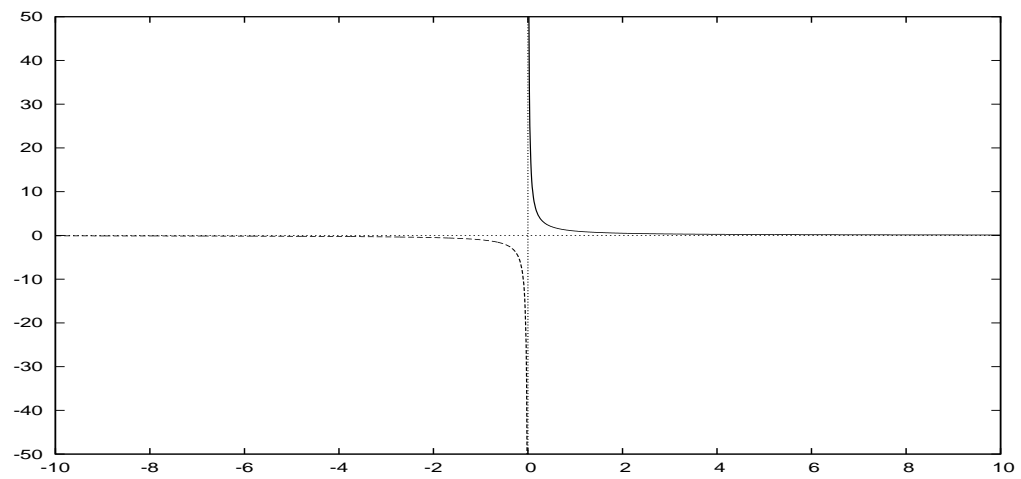
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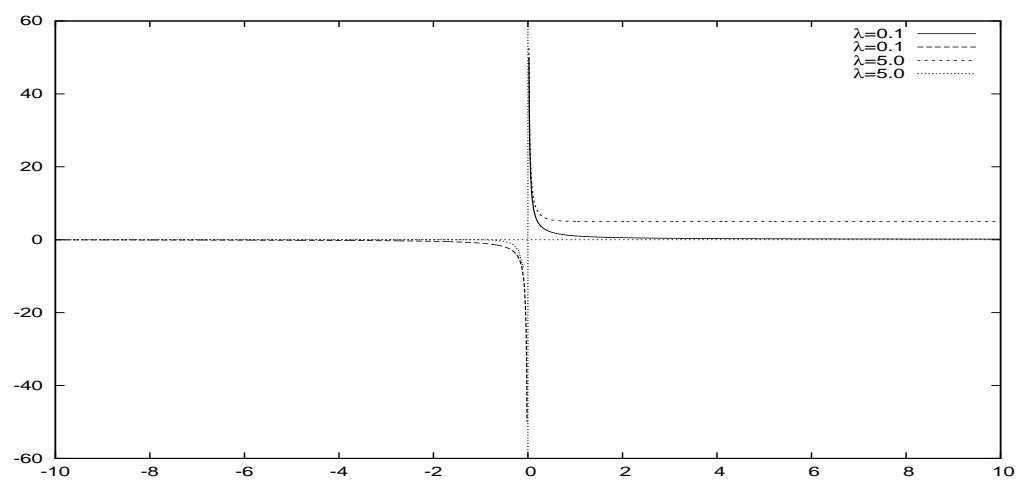
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Table 1: The energy eigenvalues of Hellmann potential.

$n$	$\ell$	$a = 1 \ b = 0.5 \ \lambda = 0.001$			$a = 1 \ b = -0.5 \ \lambda = 0.001$		
		present	Ref. [11]	Ref. [28]	present	Ref. [11]	Ref. [28]
1	0	-0.25150	-0.25100	-0.25100	-2.25050	-2.24900	-2.24900
2	0	-0.06400	-0.06349	-0.06349	-0.56300	-0.56150	-0.56150
	1	-0.06375	-0.06350	-0.06350	-0.56225	-0.56150	-0.56150
3	0	-0.02928	-0.02876	-0.02876	-0.25050	-0.24900	-0.24900
	1	-0.02917	-0.02877	-0.02877	-0.25017	-0.24900	-0.24900
	2	-0.02895	-0.02877	-0.02877	-0.24950	-0.24900	-0.24900
4	0	-0.01713	-0.01660	-0.01660	-0.14113	-0.13963	-0.13963
	1	-0.01706	-0.01660	-0.01660	-0.14094	-0.13963	-0.13963
	2	-0.01694	-0.01660	-0.01660	-0.14056	-0.13963	-0.13963
	3	-0.01675	-0.01661	-0.01660	-0.14000	-0.13963	-0.13963
$n$	$\ell$	$a = 1 \ b = 0.5 \ \lambda = 0.01$			$a = 1 \ b = -0.5 \ \lambda = 0.01$		
		present	Ref. [11]	Ref. [28]	present	Ref. [11]	Ref. [28]
1	0	-0.26502	-0.25985	-0.25985	-2.25503	-2.24000	-2.24005
2	0	-0.07760	-0.07193	-0.07193	-0.56760	-0.55270	-0.55270
	1	-0.07502	-0.07197	-0.07202	-0.56002	-0.55268	-0.55266
3	0	-0.04300	-0.03657	-0.03657	-0.25522	-0.24040	-0.24044
	1	-0.04180	-0.03661	-0.03664	-0.25180	-0.24042	-0.24040
	2	-0.03947	-0.03665	-0.03681	-0.24502	-0.24040	-0.24034
4	0	-0.03102	-0.02367	-0.02364	-0.14602	-0.13138	-0.13138
	1	-0.03031	-0.02371	-0.02371	-0.14406	-0.13137	-0.13135
	2	-0.02891	-0.02374	-0.02386	-0.14016	-0.13135	-0.13129
	3	-0.02690	-0.02378	-0.02404	-0.13440	-0.13134	-0.13120



(a) Graphical representation of  $1/x$  .



(b) Graphical representation of  $\lambda/(1 - e^{-\lambda x})$  .

Figure 1: Comparison of the functions  $1/x$  and  $\lambda/(1 - e^{-\lambda x})$ .